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1971 J. Phys. A: Gen. Phys. 4 L50

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Numerical calculation of the energy eigenvalues for a local potential

Abstract. A first-order differential equation, derived from the Schrödinger equation, is used to outline a numerical method for calculating the energy eigenvalues for a local potential.

In a recent letter (Kermode 1971) a numerical method was given for calculating the S-state energy eigenvalues for a local potential in the Schrödinger equation. This method can also be used to find the energy eigenvalues for higher angular momentum states but the range of integration would have to be increased as L , the angular momentum, increases; that is, the integration would have to be taken to a distance R such that $L(L+1)/R^2$ is negligible. This is inconvenient and it would be very useful if the centrifugal potential could be absorbed into the transformation of the logarithmic derivative of the wavefunction $u_L(r)$ (Kermode 1971) rather than be included with the short-range potential.

A suitable transformation for higher angular momentum states of the logarithmic derivative has recently been found. This leads to a very useful first-order differential equation which we now derive. The logarithmic derivative, $y_{L,\mu}(r)$, satisfies the Riccati equation

$$\frac{dy_{L,\mu}}{dr} = \mu^2 - y_{L,\mu}^2 + \frac{L(L+1)}{r^2} + V(r) \quad (1)$$

where $\mu = (-2mE/\hbar^2)^{1/2}$ for the negative energy E , m is the reduced mass and $V(r)$ is measured in units of $(\text{length})^{-2}$. The boundary condition for equation (1) is $y(0) = \infty$.

The determination of an energy eigenvalue, if one exists, is equivalent to finding a value of μ , μ_B say, such that

$$y_{L,\mu_B}(r) \underset{r \gg R}{\simeq} p_{L,\mu_B}(r) = \frac{\hat{h}_L^{(1)'}(z_B)}{\hat{h}_L^{(1)}(z_B)} \quad (2)$$

beyond the range R of the short-range potential $V(r)$, where $z_B = i\mu_B r$ and $\hat{h}_L^{(1)}(z_B)$ is the Riccati-Hankel function of the first kind (Calogero 1967). It is equal to $z_B h_L^{(1)}(z_B)$, where $h_L^{(1)}(z_B)$ is the spherical Bessel function of the third kind (Abramowitz and Stegun 1965). The prime denotes differentiation with respect to r . The expression for $p_{L,\mu}(r)$ is not as complicated as it appears; for example

$$\begin{aligned} p_{0,\mu}(r) &= -\mu \\ p_{1,\mu}(r) &= -\mu[1 + \{\mu r(1 + \mu r)\}^{-1}]. \end{aligned} \quad (3)$$

From the properties of $h_L^{(1)}(z)$, it is found that $p_{L+1,\mu}$ can be obtained from $p_{L,\mu}$ by using the formula

$$p_{L+1,\mu}(r) = -\frac{(L+1)}{r} - \frac{r\mu^2}{L+1 - rp_{L,\mu}(r)} \quad L \geq 1. \quad (4)$$

It is important to note that $p_{L,\mu}(r)$ has no singularities for $r > 0$.

A transformation for $y_{L,\mu}$ which usefully incorporates the result (2) is

$$y_{L,\mu}(r) = \mu \cot f_{L,\mu}(r) + p_{L,\mu}(r). \quad (5)$$

From equations (2) and (5) we see that, at an energy eigenvalue,

$$f_{L,\mu_B}(r) \underset{r \gg R}{=} (2n+1) \frac{\pi}{2}. \quad (6)$$

The differential equation for $f_{L,\mu}(r)$ is

$$\frac{df_{L,\mu}(r)}{dr} = -\frac{V(r)}{2\mu}(1 - \cos 2f_{L,\mu}) + \frac{\mu}{2}(1 + \cos 2f_{L,\mu}) + \sin 2f_{L,\mu} p_{L,\mu} \quad (7)$$

with the boundary condition $f_{L,\mu}(0) = 0$.

We point out that for $L = 0$ the transformation (5) and consequently the differential equation (7) are slightly different from those given previously (Kermode 1971: the numerical result given in this letter is the value of μ_B^2 and not μ_B). However, the energy dependence of $g_0(\mu) = f_{0,\mu}(R)$ is very similar *in shape* to the energy dependence of $f_0(R)$ given in the previous letter; that is, $g_0(0) = n\pi$, where n is the number of bound states or energy eigenvalues (apart from the case of a bound state at zero energy) and $g_0(\mu_B^m) = (m - \frac{1}{2})\pi$, $m \geq 1$, where $m - 1$ is the number of bound states with $\mu_B > \mu_B^m$. It was also found that $g_0(\mu) \simeq p\pi + \tan^{-1}(0.5) = p\pi + 0.46365$, where p is the number of bound states with $\mu_B > \mu$, provided that μ is not too small or too close to a bound state. In brief, $g_0(\mu)$ is practically a step function with the steps occurring at the binding energies.

A similar result was found for the higher angular momentum states. For a square-well potential of depth $V_0 \text{ fm}^{-2}$ and range $a \text{ fm}$, equation (7) was integrated numerically for various values of μ to give $g_L(\mu) = f_{L,\mu}(R)$. It was found that for a potential which has one or more bound states $g_L(\mu)$, like $g_0(\mu)$, was practically a step function. The values of the binding energies were checked against, and found to agree with, the solutions of the equation

$$\frac{\hat{j}_L'(\alpha a)}{\hat{j}_L(\alpha a)} = p_{L,\mu_B}(a) \quad (8)$$

where $\alpha = (V_0 - \mu_B^2)^{1/2}$ and $j_L(\alpha r)$ is the Riccati-Bessel function which is regular at the origin. The prime denotes differentiation with respect to r .

Unlike the case $L = 0$, there is no simple expression for the horizontal values of the steps. However, it was found in practice that the previous expression is almost valid, that is, $g_L(\mu) = p\pi + 0.46365 + \epsilon$, where ϵ is a small number ($|\epsilon| \ll 1$).

As an illustration of the use of the method, we give some of the results of our calculations in table 1. These results are for $L = 1$, $a = 1 \text{ fm}$, $R = 10 \text{ fm}$ and

Table 1

μ^2	0.01	0.2	0.2216	0.2217	0.5	1.0
$g_1(\mu)$	6.5458	6.7260	6.4224	3.7577	3.5972	3.6012
μ^2	10.0	20.0	25.0	25.2136	25.2137	30.0
$g_1(\mu)$	3.6048	3.6050	3.6051	3.6051	0.4635	0.4635

$V_0 = 40 \text{ fm}^{-2}$. From the numerical solution of the transcendental equation (8), we found that this potential has two bound states, one at 0.2218 fm^{-2} and the other at 25.2138 fm^{-2} . A fixed step length of 0.004 fm was used for the integration of the

differential equation (7). A smaller step length would give the correct answer to four decimal places. However, in general a variable step length is preferable since, for example, at the distance $\bar{r} (\simeq 0.5 \text{ fm})$ when $f_{L, \mu_B}(\bar{r})$ is equal to $\pi/2$, the slope $df_{L, \mu_B}/d\bar{r}$ is about 82 fm^{-2} for the first eigenvalue. We see from table 1 and equation (6) that the differential equation (7) enables us to calculate quickly and accurately the energy eigenvalues of a given central potential.

One of us (P.E.N.) wishes to thank the SRC for the award of a Studentship.

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KERMODE, M. W., 1971, *J. Phys. A: Gen. Phys.*, **4**, L3-5.

$K^- p \rightarrow K^+ \Xi^-$ process in the two-meson-exchange peripheral model

Abstract. The two-meson-exchange peripheral model has been used to explain the observed backward production features of the reaction $K^- p \rightarrow K^+ \Xi^-$ in the intermediate energy region.

In the past, the data on peripheral processes have been explained by extending the argument first given by Chew and Low (1959). Following them, it has been assumed that when the singularity corresponding to the exchange of the lowest mass particle is not far from the physical region then, in the part of the physical region near the pole, the peripheral diagram is dominant. Usually, this means considering one-pion-exchange diagrams. Many workers (Chan and Liu 1965, Ebel and James 1967) have extended the meaning of the word 'peripheral' to assume that the one-particle-exchange (OPE) diagram, including the baryon-exchange diagrams, would dominate the scattering. This is the basis of the OPE model. The single-baryon-exchange calculation for the peripheral process



showing backward (small u) peaking, gives differential cross sections which are orders of magnitude large and not sufficiently sharply peaked (Ebel and James 1967).

Ebel and James (1967) have applied the OPE model to the process (1) with absorption corrections. However, their prediction of the energy dependence of the total cross section is in violent disagreement with the data.